The Mathematical Marxian Theory of Capital Accumulation and the Post-Keynesian Theory of Monetary Circuit: A Synthesis

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1 Introduction

The aim of this paper is to provide a simple mathematical model in which we can analyze the accumulation of capital with the circuit of credit money. We will synthesize the two different ideas, the Marxian notion of the motion of capital and the Post-Keynesian concept of the creation of credit money, into a single model.

In section 1, a mathematical Marxian model is presented which includes debt and interest. In this model, capital is allowed to obtain the fund of accumulation not only from its own revenue but also from debt which capital needs to turn to bank to obtain. We will develop the Cambridge equation with debt ratio and interest, and establish the negative relationship between the rates of accumulation and interest. In section 2, we formulate the creation and circuit of endogenous money. We will take the three process of circuit of bank capital into consideration: creation, reserve and deposit. In the first process, credit money is created by a bank as a byproduct of loans which the bank exchanges for private bills issued by industrial capital. In the second process, the bank demands the reserve fund of cash in exchange for the bills in the money market. In the third process, credit money disappears when it is converted into cash. We will establish the characteristic equation of system in which there is a positive relationship between the interest rate and the growth rate of endogenous money supply. In section 3, we

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investigate the interaction between the accumulation of capital and the circuit of credit money. Under the assumption of the steady-state growth, the interest rate and growth rate are simultaneously determined and we will point out the possibility of crisis through the comparative dynamics.

2 Accumulation wit Debt in a Mathematical Marxian Model

In this section, we formulate a mathematical Marxian model with debt and establish the relationship between the interest rate and the growth rate.

Assume that there is an only one general commodity in an economy. One unit of the commodity is produced by a units of the commodity which is positive and less than unity[1], and l units of labor power which is also positive. The turnover period is normalized as unity. p is the nominal price level of the commodity. A profit rate is denoted by π and a nominal wage rate by w.

Now we can formulate a price relation as follows.

$$p = (1 + \pi)pa + wl.$$

Note that we assume the nominal wage w is paid at the end of the production period. If the amount of the commodity consumed by the laborer is defined to be b which is determined historically, conventionally and exogenously. $pb = w$ is assumed to be hold and b is interpreted as the real wage of laborer. It is assumed to be constant and positive.[2]

Let x be an activity level in this economy. And then px is the total sales of capital and pax is the total cost of sales. The cost of sales is also the total volume of capital owned by capitalist because the turnover period is assumed to be unity. πpax is the amount of (gross) profit. These settings are exactly the same as the linear Leontief model.[3]

[1]Otherwise, the amount of input consumed is more than the output produced under the production process. Such a situation should not be called productive but destructive. Therefore we reasonably assume that $0 < a < 1$.

[2]We also assume that there exists surplus products to reproduce the economy sustainably. Suppose the one unit of output is produced. How many amount is consumed in this economy? First, a units of input is productively consumed to produce output. Second, laborers consume bl unit of commodity to live. Therefore, the surplus products is defined to be $1 - a - bl$ which is assumed to be positive.

[3]See Morishima [13].

Now let us introduce the new specification of the concept of interest into this model (See Table 1). Marx argued in *Capital* Volume 3, chapter 23, "Interest and Profit of Enterprise," that the profit was divided into two parts; the profit of enterprise and the interest.[4] The capitalist obtains the profit πpax, but borrowing prevents her to appropriate all of her profit. Assume that the capitalist has borrowed some money from a banking sector and has to pay the interest. We denote the debt financed from the banking sector by F and the interest rate by i. iF is the interest which the capitalist has to pay back to the bank. The capitalist does not appropriate the total amount of profit, but the difference which equals the total profit minus the interest payment. We call the difference the profit of enterprise which is expressed as $\pi pax - iF$.

<table>
<tr><td colspan="3">sale
px</td></tr>
<tr><td>capital
pax</td><td colspan="2">profit
πpax</td></tr>
<tr><td>capital
pax</td><td>profit of enterprise
$\pi pax - iF$</td><td>interest
iF</td></tr>
</table>

Table 1: Contents of Sale

We define π_e as the rate of the profit of enterprise:

$$\pi_e \equiv \frac{\pi pax - iF}{pax - F}. \tag{1}$$

The numerator is the amount of the profit of enterprise, and the denominator is the owned capital which can be defined as the amount of capital minus debt. In other words, the profit rate of enterprise is the ratio of the net profit and net capital.

We can also obtain the following expression from the definition of the enterprise profit rate (1).

$$\pi = \pi_e(1 - f) + if, \tag{2}$$

[4]Marx stated as follows: "In opposition to the interest which he has to pay to the lender out of the gross profit, the remaining part of the profit which accrues to him necessarily assumes ... the form of profit of enterprise" (Marx [11], p. 496).

where f is called the debt-asset ratio. It is mathematically defined by $f \equiv F/pax$ and assumed to be constant between zero and unity. The profit rate is expressed by the weighted average of the profit rate of enterprise and the interest rate.[5]

Let us take a closer look at the content of the profit of enterprise (See Table 2). It will be divided into two parts; capitalist's consumption and saving. Assume s is the capitalist propensity to save, and then $(1 - s)(\pi pax - iF)$ is consumed by the capitalist as household and $s(\pi pax - iF)$ is saved in order to accumulate. The resource of accumulation does not consists only of this saving from the profit of enterprise, but of the new borrowing by capitalist as enterprise. Therefore,

$$pa\Delta x = s(\pi pax - iF) + \Delta F, \tag{3}$$

where Δx is the variation of activity level and ΔF is also the variation of debt level, which is the new borrowing of capitalist. The left hand side in this equation (3) represents the accumulation of capital and the right hand side represents the saving including the saving of enterprise and new borrowing. In short, this equation insists that the accumulation equals the saving from the capital enterprise and the debt.

<table>
<tr><td>saving
$s(\pi pax - iF)$</td><td>new borrowing
ΔF</td></tr>
<tr><td colspan="2">accumulation
$pa\Delta x$</td></tr>
</table>

Table 2: Saving and Accumulation

We can obtain the following expression under the assumption of the steady state growth, dividing the equation (3) by the amount of capital.

$$\pi = \frac{g}{s}(1 - f) + if, \tag{4}$$

where the growth rate is defined to be $g \equiv \Delta x/x = \Delta F/F$.

We can deduce two crucial points from this equation. The first point is that we can give the detailed expression to the rate of profit of enterprise. Comparing (2) with (4), we obtain

$$\pi_a = \frac{g}{s}.$$

[5]Lianos obtained the same result. See Lianos [9].

This is the same expression of the so-called "Cambridge equation."[6] It is here important that this formulation is not merely the definition of the profit rate of enterprise, but precisely the equation which formulates the positive linear relationship between the profit and the accumulation, extracted from a balanced condition between accumulation and saving. If there is no debt, there is no difference between the profit and the profit of enterprise, so that the equation (4) reduces to the Cambridge equation in a usual setting, in short $\pi_a = \pi$ if $f = 0$. In other words, the equation is a generalized version of the Cambridge equation which allows us to analyze the behavior of capitalists with debt. We call the equation (4) the "generalized Cambridge equation."[7]

The second point which we can deduce from the equation (4) is the negative relationship between the growth rate and the interest rate. From (4),

$$\frac{\mathrm{d}i}{\mathrm{d}g} = -\frac{1-f}{sf} < 0 \tag{5}$$

There is no difficulty to think the reason why there exists the negative relationship between the growth rate and the interest rate. Suppose the interest rate raises, then the capitalist has to pay more interest than ever before. The increment of interest payment forces her to reduce the share of the enterprise profit. The capitalist can not preserve the same amount of savings from the profit of enterprise to accumulate, and then she can not accumulate enough to keep the same pace of the economic growth with ever before.

It is noteworthy to point out that the Marxian fundamental theorem is still applicable to this setting, which insists that a capitalist can obtain a total positive profit if and only if a laborer is exploited.[8] It means that the profit rate is de-

[6] See Pasinetti [14] for a detailed explanation.

[7] See Lavoie and Godley [8] obtained the more general result. They take the stock funding into account. If the existence of share market is omitted, their "variant of Kaldor's equation"(Lavoie and Godley [8], p.289) can be reduced into the equation (4).

[8] The proof is simple. First of all, The labor value is defined to be $\lambda \equiv \lambda a + l$, which says that the value of labor λ is defined to be the sum of indirect labor λa plus direct labor l. Next, laborers are exploited if and only if $1 > \lambda b$. When a laborer provides one unit of labor, she can get b unit of real wage and obtain λb unit of labor value. If a laborer is exploited, she obtains λb less than one unit of labor which she offers. The Marxian fundamental theorem insists that the capitalist obtains the positive profit $p - pa - wl > 0$ if and only if the laborer is exploited, *i.e.* $1 - \lambda b > 0$. Suppose the profit is positive so that $p(1 - a - bl) > 0$ where $pb = w$. After some manipulation, we can obtain that $1 > l(1-a)^{-1}b$. Deduced from the definition of the labor value, $\lambda \equiv l(1-a)^{-1}$. Now we obtain $1 > \lambda b$. The Statement is just proved. This theorem is the same as the statement that profit *rate*, π, is positive if and only if the exploitation *rate*, e, is positive, that is defined to be $e \equiv (1 - \lambda b)/\lambda b$.

termined by the antagonistic relationship under production process between the capitalist and the laborer. In other words, the result of this theorem does not depends on the financial conditions, but only on the relations of production. As a result, the level of profit rate is independent from the behavior of the interest rate.

It is useful to mention why the rate of total profit and the debt-asset ratio are assumed to be constant when the interest rate is changed. The total profit rate is constant because it is, as mentioned above, determined by the exploitation of labor under the production process. The assumption of the constant debt-asset ratio may be controversial. It is natural that the debt-asset ratio should depend negatively upon the interest rate, because the capitalist tries to reduce the the debt-asset ratio if the cost of interest payment increases. In short, $f = f(i)$ and $f'(i) < 0$. But this is not controversial because there still exists the negative relationship between the growth rate and the interest rate even if we take into account the debt-asset ratio depending on the interest rate. In fact,

$$\frac{\mathrm{d}i}{\mathrm{d}g} = -\frac{1-f}{sf}\left[1-\left(\frac{g}{s}-i\right)\frac{f'}{f}\right]^{-1} < 0 \quad \text{if} \quad \frac{g}{s} > i. \tag{6}$$

Under the assumption of the steady-state growth, it is reasonable to assume that the profit rate of enterprise is greater than the interest rate. Then there still exists the negative relationship between the growth rate and the interest rate.

Summing up, We have two unknown variables derived from a Marxian model with debt; the growth rate and the interest rate. But on the other hand, we have an only one equation; the generalized Cambridge equation. We now have to leave the field of real analysis for the field of monetary analysis to look for another equation to determine the two variables.

3 A Formulation of Bank Capital

In this section, we formulate the other relationship between the growth rate and the interest rate derived from the analysis of bank behavior.[9] First, we introduce the mainstream view on the money supply founded upon the hypothesis of primary deposit and the base-multiplier process, and we also point out the controversial issues in this view. Second, we introduce the Post-Keynesian view of money supply based on the hypothesis of endogenous credit money which is created as a by-product of new loans by bank capital. Finally, we synthesize these two views to establish the relationship between the growth rate and the interest rate.

[9]Our analysis depends largely on the Itoh's works, especially Itoh and Lapavitsas[7].

3.1 Two Views of Money Supply

We now formulate the motion of bank capital based on the mainstream view. From the standpoint of the mainstream view, the bank capital behaves as "borrow low and lend high." It collects cash in deposit from household at a lower deposit interest rate. It also lends the borrowed money to industrial capitalists at a higher loan interest rate. It takes a role to mediate from surplus units to deficit units.

From the standpoint of the mainstream view, the motion of the bank capital can be formulated as follows.

$$-\mathrm{D} \longrightarrow \mathrm{M} \longrightarrow \mathrm{B} \longrightarrow \mathrm{M}' \longrightarrow -\mathrm{D}',$$

where D is deposit, M is money and B is bill. The sign of minus should be added to the deposit D, for it is not an asset but a liability owed by the bank to the depositor. Liability can be generally expressed by a negative asset, and then the minus sign should be added to the deposit of the bank.[10]

The first phase of $-\mathrm{D} \longrightarrow \mathrm{M}$ indicates that bank capital offers deposit, that is a claim to withdraw, and receives the money in which a household handed. The money and the claim are exchanged in the first phase of the motion of the bank capital.[11] In this case, the bank receives M in the form of cash as a primal deposit which can make the multiplier process start. The second phase of $\mathrm{M} \longrightarrow \mathrm{B}$ is the process of lending. The bank lends money which the household deposited at the bank in the first phase, except for the amount of reserves which she has to preserve. Capitalist borrows the money in exchange for the bill issued to start her own motion of capital. Capitalist will have to pay back the money with the loan interest rate in the next phase. The third phase $\mathrm{B} \longrightarrow \mathrm{M}$' can be called the process of collection. The bank exercises the claim to collect the money lent with the loan interest in exchange for the bill. We assume that the borrower can afford to pay money with interest, and then the bank can realize to earn the more money through lending. The final phase $\mathrm{M}' \longrightarrow -\mathrm{D}'$ is the process of the payment to the depositor if she withdraws the cash from her account in the bank. If the interest rate of deposit is lower than the one of loan, the bank can earn the profit through this total movement of capital.

This view assumes an unlikely circumstance; in this setting, the bank is lending the cash which the household deposits. First of all, nobody can lend other

[10]For the depositor, on the other hand, it is a claim to withdraw money deposited at the bank and then the minus sign should not be added because it is exactly an asset for the depositor.

[11]In other words, depositors "buy" the services of deposit in exchange for the cash and banks "sell" the services of deposit to the customers.

people's money. The bank is no exception. The bank cannot be permitted to lend the money which is deposited by the household. What if the bank lend her own money? In that case, she cannot be a bank, but should be a usurer or moneylender. Second, the bank never lend cash. Indeed the bank lends money, but it never takes in the form of the cash, but of the deposit. When the bank lends to the customer, the bank creates the account for the customer and pays to it in the form of deposit. Both the bank and the customer never touch cash in itself. Summing up, it is questionable whether the second phase M $\longrightarrow$ B is correctly formulated to establish the motion of bank capital.

This is the point for which the Post-Keynesian view criticizes the mainstream view.[12] From the standpoint of Post-Keynesian view, loans makes deposit, but primal deposit (and then the reserve) does not make deposit. When the bank lends, it mainly sets the deposit account to pay for the industrial capitalist, in exchange to the bill issued by her so that it can create credit. In other words, cash is not exchanged for the bill, but deposit is exchanged for the bill. It can be said that deposit takes the role of money. Here credit money is created.

The simplest formulation of the bank capital can be written as:

$$-\mathrm{D} \longrightarrow \mathrm{B} \longrightarrow -\mathrm{D}'.$$

The bank capital moves as $-\mathrm{D} \longrightarrow \mathrm{B}$ in the first phase of its circuit which is called loan. As mentioned above, the bank accepts the bill issued by capitalist in exchange for the deposit. In the second phase, it collects the money in the form of deposit which the capitalist pay back with the loan interest.

This formulation, however, has a blind spot. In this formula, cash never appears because it does not take any important role. This formulation implicitly assumes that the central bank unlimitedly provides the reserve to the bank. The bank never cares about how much reserve she has if reserve is always obtainable on demand. This formulation fails fully to establish what relationship exists between cash and credit.

Indeed both camps have a weak point. The mainstream view cannot grasp the centrality of the credit money, and the post-Keynesian view cannot grasp the importance of the cash. Figure 1 and 2 show where the missing links exist in each formulation. The broken lines indicate the missing link; the mainstream view fails to establish the relation between the deposit and bill, which deemed to close the possibility to analyze how credit money is created. The post-Keynesian view

[12]See Moore [12]. Our analysis of post-Keynesian credit money depends on Pollin [15]. See also Delplace and Nell [1].

$$\begin{tikzcd} & M \arrow[dl, shift left] \arrow[dr, shift left] & \\ B \arrow[ur, shift left] \arrow[rr, dash, dashed] & & -D \arrow[ul, shift left] \end{tikzcd}$$

Figure 1: mainstream View

$$\begin{tikzcd} & M \arrow[dl, dash, dashed] \arrow[dr, dash, dashed] & \\ B \arrow[rr, shift left] & & -D \arrow[ll, shift left] \end{tikzcd}$$

Figure 2: Post-Keynesian View

regards the cash as unimportant so that the category of money tends to reduce to the mere credit.

We should try to synthesize these two views.

3.2 A Modified Formulation of Bank Capital

Figure 3 shows us how to synthesize two views. The idea is simple; it is to complement the missing links in each camps. Let us explain each side of triangle in Figure 3.

$$\begin{tikzcd} & M \arrow[dl, shift left] \arrow[dr, shift left] & \\ B \arrow[ur, shift left] \arrow[rr, shift left] & & -D \arrow[ul, shift left] \arrow[ll, shift left] \end{tikzcd}$$

Figure 3: Alternative View

First, the base of the triangle indicates that the bank lend money in the form of deposit in exchange for the bill issued by capitalist and then the bank collect the loan. This process, −D ⟶ M ⟶ −D, is the result of the transaction between the bank and capitalist, and totally the same as the one of post-Keynesian view.

Now we should take the time structure into account. Suppose that the loan occurs at time t which is expressed by the arrow from −D to B. The collection will be completed at time $t + T$ where T is the period of the loan, which is expressed by the arrow form B to −D. The amount of collection at time $t + T$ should be larger than the loan at time t because the gap is interpreted as the loan interest being positive. But what we have to take into consideration is which is bigger the loan at t or the collection at t. Note that the bank collects money with the interest at time t because she would lend the money at time $t - T$. Which is larger? The answer is clear under the assumption of the steady-state growth: the amount of the loan is larger, otherwise the total stock of the bill is decreasing and then it will be impossible to expand on the exponential path. From the perspective of the *net* flows, therefore, capital moves from the deposit to the bill i.e. B $\longleftarrow$ −D. The bank just lend money in the form of deposit to obtain the bill.

Next, let us take a look at the left oblique side of the triangle. It is reasonable to interpret this process as the result of the transaction between the bank and the central bank. The bank sells the bill to the central bank if the bank needs to obtain the reserve more, whose motion is expressed by the arrow from B to M in the Figure 3. The bank buys the bill from the central bank if the bank thinks the reserve is beyond the planned amount. If we assume the ratio of reserve fund is given, the total amount of reserve, M, should increase under the steady-state growth because the loan is increasing under the steady-state path. The reserve should be preserve enough to make the constant economic growth possible. In other words, the *net* capital flow moves from B to M, i.e. B$\longrightarrow$M.

Finally, we investigate the right oblique side of the triangle. This process is reasonably interpreted as the result of the transaction between the bank and the household. The household deposits the cash in the bank account, and withdraws the cash deposited in the bank account. In other words, the cash and the deposit is exchanged in this transaction. Now we pose the question which arrow is larger. Suppose that the arrow −D $\longrightarrow$ M is larger. It means that the cash is constantly and unlimitedly moving from the household to the bank. But it is impossible because the cash which the household have at her hand is limited. It is only the central bank that can make it possible. Therefore, from the standpoint of net flow, capital moves from M to −D, i.e. M$\longrightarrow$ −D.

According to the above consideration, we can establish an alternative formulation of bank capital as follows.

$$-\mathrm{D} \longrightarrow \mathrm{B} \longrightarrow \mathrm{M} \longrightarrow -\mathrm{D}.$$

Let us briefly explain in turn. In the first phase, −D $\longrightarrow$ B, that is the transaction between the bank capital and the industrial capital. The bank lend the money in the form of deposit in exchange to the bill with discount. In the next phase, B $\longrightarrow$ M, that is the transaction between the bank and the central bank. The bank offers the bill in exchange for the cash in the money market. In the final phase, M $\longrightarrow$ −D, that is the transaction between the bank and the household. The bank pays out money if it is demanded to convert deposit into cash.

We will give a mathematical expression to this motion in the next section.[13]

3.3 A mathematical formulation of the Bank Capital

In this section, we introduce a mathematical formulation of the bank capital to argue the relationship between the growth rate and the interest rate.

3.3.1 A Diagrammatic Form of the Bank Capital

The whole motion of the bank capital is illustrated as Figure 4.

$$-D \xrightarrow[-d^{\mathrm{o}}]{+b^{\mathrm{i}}} B \xrightarrow[-b^{\mathrm{o}}]{+m^{\mathrm{i}}} M \xrightarrow[-m^{\mathrm{o}}]{+d^{\mathrm{i}}} -D$$

Phase 1 Phase 2 Phase 3

Figure 4: A Diagrammatic Form of the Bank Capital

Figure 4 is basically the same as the above-mentioned formula based on the alternative view to which some signs are added. Stock variables are represented by big letters, flow variables by small letters. The phase 1 represents the phase of loan, in which the deposit of the bank, $-D$, is transformed into the bill, B, in which the industrial capital handed. The bank makes the own debt in order to obtains the discounted bill. Let the total amount of the inflow of the discounted bill denoted by $(1 + i)^{-1}b^{\mathrm{i}}$, where i is the interest rate and b^{i} is the net bill inflow. It is exchanged for the net debt outflow, $-d^{\mathrm{o}}$. In Figure 4, the element above the arrow represents the amount of flow increasing the right-side stock, and the element below the arrow represents the amount of flow decreasing the left-side

[13] The mathematical formulation is based on the circuit of capital in some Foley's works, especially Foley [2]. Foley [2] established the behavior of industrial capital as the model of circuit of capital. But unfortunately the behavior of bank capital and the interest rate is not argued, even though the credit money supply and the rate of inflation are took into consideration. We formulate explicitly the circuit of bank capital so that we can argue the endogenous money supply.

stock. [14] The phase 2 represents the phase of the preservation of the reserve in which the bill B is transformed into cash, M. Let $-b^{\mathrm{o}}$ be the net outflow of the bill, and m^{i} be the net flow of the reserve in the cash form. In the phase 3, Cash M is transformed into the debt $-D$. We call this phase withdrawal. [15] Let $-m^{\mathrm{o}}$ be the net outflow of the cash, and the amount of the net deposit inflow is d^{i}. This process makes the deposit disappear.

We can call these relations "equivalent exchange rule." Summing up,

$$\begin{aligned} d^{\mathrm{o}}(t) &= (1+i)^{-1} b^{\mathrm{i}}(t), \\ b^{\mathrm{o}}(t) &= m^{\mathrm{i}}(t), \\ m^{\mathrm{o}}(t) &= d^{\mathrm{i}}(t). \end{aligned} \tag{7}$$

And then the circuit of bank capital will restart without stopping.[16] In next subsection we will derive mathematical expressions of bank capital circuit from Figure 4.

3.3.2 A Mathematical Formula of the Bank Capital

First of all, we investigate how the stocks of bank capital increase in the circuit. The stocks in the circuit, $-D$, B and M, are governed by the following rule:

$$\begin{aligned} \dot{B}(t) &= (1+i)^{-1} \left(b^{\mathrm{i}}(t) - b^{\mathrm{o}}(t) \right), \\ \dot{M}(t) &= m^{\mathrm{i}}(t) - m^{\mathrm{o}}(t), \\ -\dot{D}(t) &= d^{\mathrm{i}}(t) - d^{\mathrm{o}}(t). \end{aligned} \tag{8}$$

These equations (8) follow from the book-keeping rule: that is, each stock is increased by the inflow and decreased by the outflow, and then each stock is accumulated when the inflow is more than the outflow. Note that the stock of the bill is assumed to be evaluated at acquisition cost basis.

[14] Note that the debt in the balance sheet of the bank capital is the quantity of minus, because it is not an asset, but a liability. The negative number minus the negative number equal the negative number, and the absolute value is increasing.

[15] We ignore saving accounts, and assume that there are only checking accounts open. And then we can ignore the deposit interest rate.

[16] This model implicitly assumes that bankers accumulate all profit to lend, and are not remunerated so as to consume nothing. In other words, the banking sector is a sort of "accumulation machine." Let us explain this assumption with successive motions of bank capital, such as $-\mathrm{D}_1 \longrightarrow \mathrm{M}_1 \longrightarrow -\mathrm{D}'_1 \cdot -\mathrm{D}_2 \longrightarrow \mathrm{B}_2 \longrightarrow -\mathrm{D}'_2 \cdots\cdots$. In this context, the assumption of "accumulation machine" is represented by $\mathrm{D}'_1 = \mathrm{D}_2$. In the equations (7), if we take the banker's remuneration as household into account, the first equation of (7) should be changed into $d^{\mathrm{o}}(t) = (1+i)^{-1} b^{\mathrm{i}}(t) + r(t)$, where $r(t)$ is banker's remuneration as household at time t.

Now we formulate flow variables. The inflow and outflow are related by the convolution:

$$\begin{aligned} b^{\mathrm{o}}(t) &= \int_{-\infty}^{t} b^{\mathrm{i}}(t')\beta(t-t')\,\mathrm{d}t', \\ m^{\mathrm{o}}(t) &= \int_{-\infty}^{t} m^{\mathrm{i}}(t')\mu(t-t')\,\mathrm{d}t', \\ d^{\mathrm{i}}(t) &= \int_{-\infty}^{t} d^{\mathrm{o}}(t')\delta(t-t')\,\mathrm{d}t'. \end{aligned} \tag{9}$$

β represents a distributed lag in the transition from phase 1 to 2 in Figure 4, interpreted as the proportion of bill inflow at time t, that are sold at time $t+t'$. μ is a distributed lag in the transition from phase 2 to 3, interpreted as the proportion of cash inflow at time t, that are withdrawn at time $t+t'$. δ is also a distributed lag in the transition from the birth to death of the bank deposit, which is also interpreted as the proportion of debt outflow set by loan at time t, which are disappeared to be withdrawn at time $t+t'$. β, μ and δ are nonnegative and integrate to unity over the positive half-line.

Under the stationary state, the initial conditions must satisfy the following equations.[17] Reasoning from three equations (9),

$$\begin{aligned} b^{\mathrm{o}} &= b^{\mathrm{i}}\beta^*(g), \\ m^{\mathrm{o}} &= m^{\mathrm{i}}\mu^*(g), \\ d^{\mathrm{i}} &= d^{\mathrm{o}}\delta^*(g), \end{aligned} \tag{10}$$

where

$$\beta^*(g) = \int_0^{\infty} \beta(t)\exp(-gt)\,dt$$

which is the Laplace transform of the lag function $\beta(.)$ and similarly for $\mu^*(g)$ and $\delta^*(g)$. The Laplace transform has specific properties as follows: $\beta^*(0) = 1$, $\mathrm{d}\beta^*(g)/\mathrm{d}g < 0$, $\lim_{g\to\infty}\beta^*(g) = 0$.

From these equations, we can drive the stock variables: B, M, and $-D$. Noting that all stock variables grow at the rate of g, and substituting (10) to (8), we get:

$$\begin{aligned} B &= b\left(\beta^*(g)^{-1} - 1\right)g^{-1}, \\ M &= b\left(\delta^*(g)\beta^*(g)^{-1}\mu^*(g)^{-1} - \delta^*(g)\beta^*(g)^{-1}\right)g^{-1}, \\ -D &= b\left(\delta^*(g)\beta^*(g)^{-1} - \beta^*(g)^{-1}\right)g^{-1}, \end{aligned} \tag{11}$$

[17] We omit the initial time subscript such as $x(0) = x$.

where $b \equiv b^o(1+i)^{-1}$.

We should set some assumptions about the level of stock variables.

Assumption 1 The following relations are assumed.

1. Credit creation occurs.

$$M < D \quad \Longleftrightarrow \quad \mu^*(g)^{-1} < \delta^*(g)^{-1}. \tag{12}$$

2. Loan makes deposit.

$$D < B \quad \Longleftrightarrow \quad \delta^*(g)^{-1} < \beta^*(g)^{-1}. \tag{13}$$

The first assumption is the same as the following statement; the money multiplier should be more than unity. The second assumption is easily understood if we imagine the opposite case. Suppose that $D > B$, then $D - B$ is lend from the reserves. But the cash is not lent directly to the capitalist. This contradiction occurs because the supposition $D > B$ is assumed. Therefore $D < B$ is reasonable if loan makes deposit.

3.3.3 The Interest-Growth Relation in the Circuit of Bank Capital

From the system of equations (7) and (10), we get the following the characteristic equation.

$$i = \frac{\delta^*(g)}{\beta^*(g)\mu^*(g)} - 1. \tag{14}$$

We can extract some specific characters from this equation under the assumption 1.

Proposition 1 Equation has the following characters.

1. No growth, no interest.

$$i \to 0 \quad \text{if} \quad g \to 0 \tag{15}$$

2. Infinite growth, infinite interest.

$$i \to \infty \quad \text{if} \quad g \to \infty \tag{16}$$

3. positive growth, positive interest.

$$i > 0 \quad \text{if} \quad g > 0 \tag{17}$$

4. The interest rate is a continuous function of the growth rate.

From these propositions, it is natural to assume that the interest rate increases if and only if the growth rate increases. Roughly speaking, if the interest rate increases, it is easy to accumulate because the profit of bank increases.

Assumption 2 There exists positive relationship between the growth rate and the interest rate if we assume the following condition.

$$\frac{\mathrm{d}i}{\mathrm{d}g} > 0 \quad \text{if} \quad \frac{\mathrm{d}\delta^*(g)\beta^*(g)^{-1}\mu^*(g)^{-1}}{\mathrm{d}g} > 0. \tag{18}$$

Finally we obtain the interest-growth relation through the behavior of bank capital. There exists the positive relationship between the growth rate and the interest rate.

4 The interaction between the real and monetary economy

The interest rate and the growth rate are simultaneously determined as the Figure 5 shows. The downward sloping line indicates the generalized Cambridge equation in the real economy and the upward sloping curve represents the characteristic equation in the monetary economy. We can obtain some interesting features from the comparative dynamics.

4.1 Exploitation and Profit Squeeze

Suppose that the workers are exploited more than ever before because capitalists obtain new technology. It means that the ratio of exploitation is increased and then the profit rate is increased (because the fundamental Marxian theorem holds). In the figure 5, it causes a upward shift of the generalized Cambridge equation. Therefore,

$$e \uparrow \longrightarrow \pi \uparrow \longrightarrow i \uparrow \wedge \ g \uparrow$$

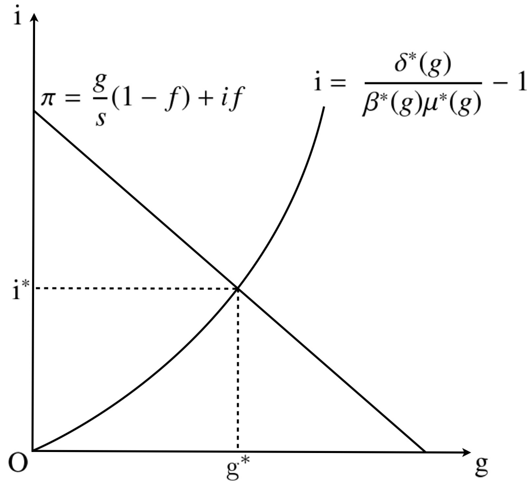

Figure 5: The Determination of the Interest Rate and the Growth Rate

Capitalists can enjoy an economic boom under this system.

The profit squeeze can be interpreted as the opposite case of the above. If the real wage gradually increases under an economic boom, the profit is also gradually squeezed.

$$b \uparrow \longrightarrow \pi \downarrow \longrightarrow i \downarrow \wedge \ g \downarrow$$

It would be noteworthy that the value of the interest rate decreases under the process of the profit squeeze. In general, on the contrary, the interest rate would be increased at the end of booming because the speculations occurs. It seems contradict between this model and general case. But it is possible that they are perfectly compatible. It needs lots of fund to make the speculation accelerated. Where does it come from? The answer is clear; it comes from inside the circuit of bank capital. The bank will lend more than before so that the credit money will be created. It means that the B should be increased for all g. It is possible if the value of $\beta^*(g)$ is decreased for all g. Let us formulate $\beta^*(g)$ to analyze more concretely.

Assume the all process of bank capital is constituted as the manner of "first-in, first-out" which makes each process of bank capital reduced to the simple time delay. If we define the time delay of loan as T_B, reserve as T_M and deposit as T_D respectively, we get:

$$\beta^*(g) = \exp(-gT_B),$$
$$\mu^*(g) = \exp(-gT_M),$$
$$\delta^*(g) = \exp(-gT_D).$$

We obtain the following concrete characteristic function from these formulations.

$$i = \exp g(T_B + T_M - T_D) - 1. \tag{19}$$

And instead of assumption 2, we have to assume

$$T_B + T_M - T_D > 0. \tag{20}$$

The amount of the loan can be increased if T_B is larger. It makes the characteristic function shift upward. If the effect of this shift is bigger than that of the shift of the generalized Cambridge equation, the interest rate will be increased. The interest rate can be increased at the end of the economic boom if the change of T_B makes the characteristic function shift enough large to increase the interest rate.

4.2 Monetarist and Post-Keynesian approach to the financial process

First of all, we formulate the idea of the monetarist in this model. According to the monetarist, the central bank can and should increase the money supply by a constant percentage rate, k.[18] What makes this policy possible? It depends largely (and implicitly) on the flexible adjustment of the time period T_M. The central bank intervenes the money market and adjusts the time period of reserve T_M to keep the monetary growth rate constant. In short,

$$g \text{ is given} \longrightarrow T_M \longrightarrow i.$$

We have two equations so that we should have only two variables in order fully to close the model. In this case, g given exogenously, the endogenous variables are i and T_M.

Second, we can formulate the idea of Horizontalist which criticizes that of the monetarist.[19] The central bank would set the level of the interest rate, not the growth rate of money supply, for it is not a price taker and quantity setter but a price setter and quantity taker. And the central bank never refuse the request of reserve from commercial banks. The bank should change the T_M to preserve enough reserve to keep the interest rate which the central bank assigned.

$$i \text{ is given} \longrightarrow T_M \longrightarrow g.$$

[18] See Friedman [3] for his "k-percent rule".

[19] See Moore [12].

As we have just seen, the causality is clear contrary between Horizontalist and monetarist. At the same time, however, they have the common feature in those settings; the central bank can set one of the variables exogenously. Monetarist insist that the growth rate of money is exogenously determined and Horizontalist insists that the interest rate is exogenously determined. The causality is totally different but the exogeneity is absolutely same.

We can easily imagine another camp which would insist these variables are endogenously determined — that is structuralist.[20] Structuralist insists that the interest rate and the growth rate of money endogenously determined. And more importantly, structuralist point out that bank tries to innovate the technology of finance. In this model, the innovation of finance accelerates the liquidity of financial assets. It makes the turnover rate of financial stock variables, especially B, much higher. It means that $1/T_B$ get higher. This change makes the characteristic function shift rightward.

$$1/T_B \uparrow \longrightarrow i \downarrow \longrightarrow g \uparrow$$

The financial innovation enhances the economic expansion: there exists the economic regime of "high-growth, low-interest."[21] Capitalist and banker seem to enjoy a permanent economic boom if the financial innovation constantly occurs. But this does not means that the economic crisis never happen. We will point out the possibility of crisis in the concluding remarks.

Concluding Remarks

We can find the possibility of crisis in the process of the circuit of the bank capital.

First of all we assume each net flow as positive. But this assumption depends largely on the assumption of the steady-state growth such as the path on the

[20] See Pollin [15]. And also see Deleplace and Nell [1].

[21] We can provide the contrary example. If we assume that bankers obtain their remuneration as household and then consume in a commodity market, the regime of "low-growth, high-interest" holds. In the figure 5, the new regime makes the characteristic function shift left side because the decrement of saving to lend makes the growth rate of money supply decrease. In short,

$$\text{Banker's remuneration to consume} \uparrow \longrightarrow \text{Banker's savings to lend} \downarrow \longrightarrow i \uparrow \wedge\ g \downarrow$$

It may be doubtful if banker's consumption reduces the amount of credit money, but the attitude of banker's savings to lend has an clear effect to the interest rate and the growth rate of money supply.

schema of the expanded reproduction in *Capital* Volume 2. But it is not always guaranteed. For example, if the bank cannot find any company to want to borrow the large amount of money, the amount of the loan outflow at time t is smaller than the amount of the collection inflow at time t. It makes the absolute volume of the stock of the bill and the deposit decreased. If this economic circumstance continues, the economy would shrink and the growth rate might would be negative. In other words, we can not obtain the characteristic equation which maintain the positive relationship between the interest rate and the growth rate. It holds only under the assumption of the steady-state growth, otherwise the economy would face to negative growth rate under depression or crisis.

We can also point out the possibility of the crisis even though each net flow is positive. In this economy, there is a positive relation between the interest rate and the growth rate in the monetary field. But this relation is obtained by assuming the condition of (18) or (20). If this condition is violated, we cannot obtain the positive relationship. In this case, the growth rate would be negative even though the interest rate is still positive.

Let us take a look at the each phase of bank circuit to investigate whether crisis can happen. First, crisis can happen if η_δ or T_D is sufficiently-large. It means that economic agents extraordinarily withdraw their deposit which makes the reserve of bank capital drastically reduced. It would trigger a run on the bank. This is a typical story of the monetary crises. Second, crisis can happen if η_β or T_B is sufficiently-small. It means that the downturn of the growth rate should make banks reluctant to issue new loans, because banks can not find healthy business partners to whom banks want to lend. Consequently, credit money is not created. We call the collapse of η_β or T_B the failure of "loan-creation." Third, crisis can happen if η_μ or T_M is sufficiently-small. This case may rarely happen under the modern system of the central bank, but it can happen if banks cannot obtain enough cash in a money market. In this case, there exists a sort of "realization-failure" from the form of cash to credit. These failure of the metamorphosis of bank capital implies that the crisis can happen.

Our approach can shed light on the possibility of crisis, but it has also some weak points which should be relaxed. This approach should incorporate the inflation rate and also clear the adjustment process of inflation rate. In other words, we should distinguish the nominal value from the real value. This weakness is fundamentally connected with another weakness. It concentrates on the analytical attention to the steady state growth. The assumption of the steady state growth allows us to omit any adjustment process in this economy. Mathematically, we can concentrate on the absolute value of endogenous variable, such as g, but we should incorporate how the endogenous variables are adjusted and how the change of endogenous variables, say $\dot{g}$, is determined. The omission of the analysis of adjustment eliminates the analytical domain of the instability of capitalism. In other words, we can only point out the possibility of crisis, not the reality of crisis.

The future direction of this study will be the one that encompasses these topics.

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